Int. J. Heat Mass Transfer. Vol. 9, pp. 1471–1488. Pergamon Press 1966. Printed in Great Britain

AN INTEGRAL METHOD OF SOLUTION OF THE GENERAL HEAT AND MASS TRANSFER PROBLEM

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(*Received* 10 *December* 1964)

Аннотация—В работе предлагается метод решения нестационарных линейных задач переноса в предположении, что известно решение задачи о переносе импульса для неограниченного пространства. Рассматриваются различные краевые задачи для широкого класса областей, составляется и строится интегральное уравнение, решением которого является функпия влияния для данной области при адиабатически заизолированной границе. В работе показано, что построенное интегральное уравнение может быть решено методом последовательных приближений, исследована быстрота сходимости. Кроме того, показано, что члены ряда последовательных приближений допускают весьма ясную и наглядную физическую интерпретацию, позволяющую истолковать процесс переноса.

Построение функции влияния дает возможность указать в самом общем виде метод решения первой краевой задачи. Для этого с помощью функции влияния для данной области строится новое интегральное уравнение для отыскания теплового потока, осуществляющего процесс переноса при заданных граничных условиях задачи. Это интегральное уравнение вновь решается методом последовательных приближении, также допускающих весьма наглядную физическую трактовку.

NOMENCLATURE

a^2, thermal diffusivity coefficient;
c, diffusion rate;
C, specific heat;
D, region;
$f(Q)$, initial temperature;
$F(\mathscr{P}, t)$, function of sources;
$G(Q, \mathscr{P}, t)$, temperature field, Green function for region D;
J, heat content;
k, $= C\gamma$;
L_t, Laplace transformation;
L, M, N, O, P, Q, points;
$n = 1, 2, 3$, reflection number;
P, differential operator;
q, heat flux;
r, radius-vector length $|QP|$;
R, space;
S, boundary surface of the region D;
t, time;
T, temperature;
x, y, z, co-ordinates.

Greek symbols

α, heat-transfer coefficient;
γ, specific weight;
Γ, Green function for whole space;
δ, delta-function;
Δ, Laplace operator;
ε, emissivity;
ξ, time moment;
η, co-ordinate;
ρ, φ, θ, spherical co-ordinates;
λ, heat conductivity;
ν, external normal;
τ, time moment;
$\underline{\varphi}$, temperature of body surface;
$\overline{\varphi}(\theta, \varphi, M)$, dissipation function.

1. RECENTLY a number of works have appeared

in which a diffusion (heat-conduction) equation is stated to include the effect of the finite diffusion rate of a substance or energy carrier, e.g. the papers by Fock [1] dealing with one-dimensional diffusion of a light beam, by Davydov [2], Lyapin [3] and Monin [4] on turbulent diffusion, by Kramer and Chandrasekhar (see reference [6]) who thoroughly analysed the Focker–Planck difference equation describing the probability of the presence of a particle in Brownian motion at the point x and time instant t, by Goldstein [5] who solved the problem of random motion of particles which lost their "memory" (i.e. particles which are in Markovian process), by Davies [6], Vernotte [7], Cattaneo [8] and others.

In these works a hyperbolic equation, the so-called telegraph equation, is obtained by different ways for the unsteady-state diffusion (heat conduction) process. The equation differs from an ordinary parabolic one by the presence of the term $1/c^2\ \partial^2 T/\partial t^2$ and is the result of a more profound analysis of the phenomenon. In the above term c is the diffusion rate. It has been proved that the classical approximation is not applicable to a number of problems, namely to those in which the diffusion rate cannot be assumed infinite or the mean free path of particles negligibly small.

Analysis of the Boltzmann equation [9] shows that a similar pattern exists for example in metals at high temperature gradients. In this case there is no classical relationship between heat flux and gradient. In particular, infinite gradients do not cause infinite fluxes, which in reference [9] is referred to as paradox in heat conduction. It means that the basic solution of a heat-conduction equation for short time intervals (in the region of large gradients) does not correspond to the true temperature field.

In reference [6] the basic solution (for the whole space) of the telegraph equation is given and it is proved that for long time intervals it asymptotically approaches the basic solution of the heat-conduction equation (of parabolic type).

The influence function for two- and three-dimensional space is given in reference [10]. In the latter case the function is of the form

$$\Gamma(Q, \mathscr{P}, t) = c/r\, e^{-\frac{1}{2}a^2c^2t}\Big\{\delta(ct - r) + \frac{a^2cr}{2\sqrt{r^2 - c^2t^2}} \times J_1\left[\tfrac{1}{2}a^2c\sqrt{r^2 - c^2t^2}\right]u(ct - r)\Big\},$$

where a^2 is the thermal diffusivity coefficient, c is the diffusion rate (or the rate of thermal excitation transfer), $r = |Q\mathscr{P}|$, J_1 is the Bessel function and

$$u(\eta) = \begin{cases} 0 & \text{at} \quad \eta < 0 \\ 1 & \text{at} \quad \eta > 0 \end{cases}$$

when

$$c \to \infty, \qquad \Gamma(Q, \mathscr{P}, t) \to \frac{1}{(2a\sqrt{\pi t})^3}\exp\left[-r^2/4a^2t\right].$$

This paper presents the construction of the general solution of a wide range of transient non-linear transfer problems, with any initial and boundary conditions, on the basis of both the function of the source in the form of unit pulse, determined over the whole space R and obtained analytically or by the data of one fundamental experiment, and the superposition method which gives rise to no doubts as far as low-temperature fields are considered. It does not resort to any additional suppositions on the character of the heat-conduction and diffusion processes.

This method, called an integral one, allows for both the inertness of the process and the velocity of heat or substance propagation and includes, as a particular case, the solution of the same problem based on the Fourier hypothesis, thus being more extensive than the one based on the solution of some differential equation. Henceforth the transfer process is referred to as heat conduction, and talk of heat propagation.

The most general heat-conduction problem is stated as follows: Find the temperature field in the region D (which, in general, is not singlyconnected) if it has some initial temperature distribution $T(\mathscr{P}, t)|_{t=0}$ and heat is liberated at each point of the region with intensity $F(\mathscr{P}, t)$. We know the temperature of the body surface $\varphi(S, t)$ (the boundary condition of the first kind) or the value of the heat flux on the boundary S of the body $q(S, t)$ (the boundary condition of the second kind), or alternatively the heat transfer law and ambient temperature are given, e.g. Newton's law of cooling $q = \alpha(T_{\text{sur}} - T_{\text{bound}})$ (the boundary conditions of the third kind). In addition, boundary conditions of the fourth kind may be given for the surface S. This is the so-called contact heat conduction, when the region D is in contact (i.e. has a common boundary) with another region D^* of definite physical properties. A close contact provides continuity of the temperature field at a contact point. By the heat conservation law, the flux to the region D is the flux to D^* with an opposite sign.

2. Let D be a convex region with the boundary S. The problem is to estimate the temperature field $G(Q, \mathscr{P}, t)$ at point $\mathscr{P}$ and time t, when a unit amount of heat is liberated instantaneously at some point Q of the region at $t = 0$, and the surface is thermally insulated. In solving this problem, the principle of superposition is supposed to be valid and the influence function $\Gamma(Q, \mathscr{P}, t)$ is considered known for the whole space $R(Q, \mathscr{P} \in R)$. First of all consider the following problem: at $t = 0$ a unit heat pulse is liberated at point Q of three-dimensional space R. Let us encircle the point Q by a closed convex surface S which does not obstruct heat propagation. Find the heat flux $q(Q, M, t)$ at point M of the surface in the direction of the external normal v. Consider the sphere Ω with the radius $z = |QM|$ and point Q as centre. The heat flux q_Ω through its surface, being directed out of Ω, is equal to the time derivative of the enthalpy in the region $R - \Omega$.

$$q_\Omega(Q, M, t) = \frac{\partial}{\partial t} \iiint\limits_{R-\Omega} k\Gamma(Q, \mathscr{P}, t)\, \mathrm{d}\mathscr{P}$$

where k is the product of the specific heat by the specific weight. Owing to the isotropy of the space both the heat flux and the temperature field $\Gamma(Q, \mathscr{P}, t)$ possess spherical symmetry relative to the point where the heat pulse appears. This means that on the sphere with the point Q as the centre the flux q_r does not change its scalar value. q_r is a flux through a point on the sphere in the direction of $r = |Q\mathscr{P}|$. Therefore

$$q_\Omega = \iint\limits_{S_\Omega} q_r \,\mathrm{d}S_\Omega = q_r \,.\, 4\pi r^2$$

and

$$q_r = k \frac{1}{r^2} \int\limits_{r=|QM|}^{\infty} \rho^2 \frac{\partial \Gamma(\rho, t)}{\partial t} \,\mathrm{d}\rho.$$

Let $\mathrm{d}Q_1$ be the heat flux through the element $\mathrm{d}S_\Omega$ of the spherical surface for the time $\mathrm{d}t$:

$$\mathrm{d}Q_1 = q_r \,\mathrm{d}S_\Omega \quad \mathrm{d}t,$$

and $\mathrm{d}Q_2$ is the heat flux through the element $\mathrm{d}S$ of an arbitrary surface S:

$$\mathrm{d}Q_2 = q_v \,\mathrm{d}S \,\mathrm{d}t.$$

$\mathrm{d}S_\Omega$ and $\mathrm{d}S$ are in the vicinity of the point M. Using $\mathrm{d}Q_1 = \mathrm{d}Q_2$ and $\mathrm{d}S = \mathrm{d}S_\Omega/\cos(r, v)$ one obtains $q_v = q_r \cos(r, v)$. Then

$$q_v = k \frac{\cos(r, v)}{r^2} \int\limits_{r=|QM|}^{\infty} \rho^2 \frac{\partial \Gamma(\rho, t)}{\partial t} \,\mathrm{d}\rho.$$

But

$$\frac{\partial}{\partial t} \iiint\limits_{R-\Omega} k\Gamma \,\mathrm{d}V = -\frac{\partial}{\partial t} \iiint\limits_{\Omega} k\Gamma \,\mathrm{d}V.$$

With this in mind one can write q_v in the form

$$q_v = -k \frac{\cos(r, v)}{r^2} \int\limits_0^{r=|QM|} \rho^2 \frac{\partial \Gamma(\rho, t)}{\partial t} \,\mathrm{d}\rho. \quad (1)$$

In case of two-dimensional space

$$q_v = -k \frac{\cos(r, v)}{r} \int\limits_0^{z} \rho \frac{\partial \Gamma(\rho, t)}{\partial t} \,\mathrm{d}\rho.$$

It is not difficult to show that the function $q(Q, M, t)$ is a continuous variable of Q and M for $t > 0$. Moreover it will be shown later that q_v is singular at $r \to 0$ and $t \to 0$. The amount of heat through S for the time t is equal to

$$\int_0^t \mathrm{d}\tau \int_S\int q_v \,\mathrm{d}S.$$

It is also the difference between the enthalpy of the region $R - D$ at the time moment t and that at the initial moment. Thus

$$\int_0^t \mathrm{d}\tau \int_S\int q_v(Q, M, \tau)\,\mathrm{d}M = \int\int_{R-D}\int k\Gamma(Q, \mathscr{P}, t)\,\mathrm{d}\mathscr{P} = 1 - \int\int_D\int k\Gamma(Q, \mathscr{P}, t)\,\mathrm{d}\mathscr{P}. \quad (2)$$

Hence

$$\int_0^t \mathrm{d}\tau \int_S\int q_v \,\mathrm{d}S < 1 \quad (3)$$

for any point $Q \in D$ and for all finite t. Note that q_v is a positive function for convex regions since $q_r > 0$ and $\cos(r, v) > 0$ for such regions.

Let us suppose now that the boundary S of the region D is mobile, and the region D at all $t \in (O, t_0)$ is convex. It is also supposed that the motion of the boundary S does not disturb the heat conduction process in R, i.e. S is "transparent" for heat. The change of heat content in the region D is provided both by heat conduction and by displacement of the boundary. So that

$$\frac{\partial J(Q, t)}{\partial t} = k\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta \int_0^{r_0(\theta,\varphi,t)} \rho^2\frac{\partial\Gamma(\rho, \theta, \varphi, t)}{\partial t}\,\mathrm{d}\rho + k\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\cdot r_0'(\theta, \varphi, t)\cdot r_0^2(\theta, \varphi, t) \times \Gamma[r_0(\theta, \varphi, t), \theta, \varphi, t]\,\mathrm{d}\theta,\dagger$$

† Since in general the origin of co-ordinates does not coincide with the point of heat pulse onset, $\Gamma(Q, \mathscr{P}, t)$ is not a spherically symmetrical function relative to the point O but depends also on θ and $\varphi(\rho = |O\mathscr{P}|)$.

where $r_0'(\theta, \varphi, t) = |OM(t)|$,

$$r_0'(\theta, \varphi, t) = \frac{\mathrm{d}r_0(\theta, \varphi, t)}{\mathrm{d}t}$$

and the point O is the origin of co-ordinates. The first summand describes the change of enthalpy by heat conduction, the second one that due to the change of the region D itself. In this case therefore

$$q_v(Q, M, t) = -k\frac{\cos[r(t), v(t)]}{[r(t)]^2} \times \int_0^{r(t)} \frac{\partial\Gamma(\rho, t)}{\partial t}\rho^2\,\mathrm{d}\rho - k\cos[r_0(t), v(t)] \quad (4)$$

$$r_0'(t)\Gamma[Q, M(t), t]$$

where $r(t) = |QM(t)|$. If the moment τ of the origin of the heat pulse does not coincide with the time origin $(O \leqslant \tau \leqslant t)$, then

$$\frac{\partial\Gamma(\rho, t-\tau)}{\partial t}$$

and $\Gamma[Q, M(t), t - \tau]$ should be substituted for

$$\frac{\partial\Gamma(\rho, t)}{\partial t}$$

and $\Gamma[Q, M(t), t]$ respectively in the expression for q_v.

3. The principle of superposition, as can be easily shown, makes it possible to write the solution of the general heat conduction problem for boundary conditions of the second kind directly, if the influence function G is known for a given region. For example, when $T|_{t=0} = f(\mathscr{P})$, the heat flux on the surface equals $q(M, t)$ and the function $F(\mathscr{P}, t)$ of the source acting in D is given, the temperature $T(\mathscr{P}, t)$ can be found from the expression

$$T(\mathscr{P}, t) = k\int\int_D\int f(Q)G(Q, \mathscr{P}, t)\,\mathrm{d}Q + \int_0^t\mathrm{d}\tau\int_S\int q(M, \tau)\cdot G(M, \mathscr{P}, t-\tau)\,\mathrm{d}M + \int_0^t\mathrm{d}\tau\int\int_D\int F(Q, \tau)\cdot G(Q, \mathscr{P}, t-\tau)\,\mathrm{d}Q. \quad (5)$$

Here the influence of the initial conditions on the temperature field is written for the case when these conditions are specified by a given field function at an initial instant in time. For transfer problems with initial conditions not completely defined by this function alone, the effect of the initial conditions will, of course, be different. If, for instance, the transfer process described by the hyperbolic equation is discussed, then it is necessary to add terms to the right-hand side of the expression

$$\frac{1}{c^2}\iiint\limits_D \left[\frac{\partial T(Q,t)}{\partial t}\bigg|_{t=0} G(Q,P,t) + f(Q)\frac{\partial G(Q,P,t)}{\partial t}\right] dQ.$$

The expression for the effect of the boundary conditions and inner sources is unchanged. Therefore the basic statements and conclusions of the assumed method are valid in this case.

Thus determination of the function G is the primary problem in the study of heat conduction.

By considering the mechanism of heat (mass) transfer one can conclude that the original heat pulse is propagating in the region D in the same way as in the whole space. Due to the superposition of heat insulation upon the boundary, this pulse leaves a heat charge $q_v(Q, M, \tau)\mathrm{d}S\,\mathrm{d}\tau$, being called the first reflection in this paper, on each element $\mathrm{d}S$ of the surface for the time $\mathrm{d}\tau$. This charge will, in its turn, propagate by the principle of independence of action as it does in the space R, leaving the charge

$$q_v(Q, M, \tau)\,\mathrm{d}S\,\mathrm{d}\tau\,.\,q_v(M, N, \xi - \tau)\,\mathrm{d}\sigma\,\mathrm{d}\xi$$

on the elements of the boundary $\mathrm{d}\sigma$ at the moment ξ for the time $\mathrm{d}\xi$. ($M, N \in S; \tau \leqslant \xi \leqslant t$). This result of the first reflection effect on the boundary elements is called the second reflection, etc. The task is to describe the process as an infinte series of reflections allowing for the condition of absolute "opacity" for heat of the boundary S and to prove its convergence. Let us find, for example, the contribution of the second reflections (the third term of the series) to the temperature field G. For this purpose multiply the value of the second reflection by the influence function for the whole space

$$\Gamma(N, \mathscr{P}, t - \xi)$$

and integrate along the surface by the variable N and time ξ, changing from τ to t. This means that for the time $t - \tau$ the first reflection leaves heat charges at any point N of the surface. Then changing from O to t. This means that the first reflections, which give rise to the second ones and the infleunce of which is being allowed for now, appear at any point M of the surface for the time t. Consequently, the appropriate addition is of the form

$$\int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau)\,\mathrm{d}M \int_\tau^t \mathrm{d}\xi \iint_S q_v(M, N, \xi - \tau)\,.\,\Gamma(N, \mathscr{P}, t - \xi)\,\mathrm{d}N,$$

and the unknown series is

$$G(Q, \mathscr{P}, t) = \Gamma(Q, \mathscr{P}, t) + \int_0^t \mathrm{d}\tau \iint_S q_v(\theta, M, \tau)\,.\,\Gamma(M, \mathscr{P}, t - \tau)\,\mathrm{d}M$$
$$+ \int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau)\,\mathrm{d}M \int_\tau^t \mathrm{d}\xi \iint_S q_v(M, N, \xi - \tau)\,.\,\Gamma(N, \mathscr{P}, t - \xi)\,\mathrm{d}N + \ldots. \quad (6)$$

Note that the reflection series is very similar to the solution of an integral equation with the kernel q_v, which has been obtained by the method of successive approximations. The integral equation can be constructed, based on the fact that the effect of the adiabatically insulated boundary (heat barrier) will result in the "reflection" of the flux $q_v(Q, M, \tau)$ into the region D. Otherwise, the function G is the result of the superposition of heat sources with the intensity q_v, distributed along S plus the source function for an infinite space

$$G(Q, \mathscr{P}, t) = \int_0^t d\tau \int\int_S q_v(Q, M, \tau) \times G(M, \mathscr{P}, t - \tau)\, dM + \Gamma(Q, \mathscr{P}, t)$$

or

$$G(Q, \mathscr{P}, t) = \int_0^t d\tau \int\int_S q_v(Q, M, t - \tau) \times G(M, \mathscr{P}, \tau)\, dM + \Gamma(Q, \mathscr{P}, t). \quad (7)$$

In a particular case when Γ and G are Green functions of the heat conduction equation with boundary conditions of the second kind for the whole space and the region D respectively, then equation (7) is directly obtained by applying the Green formula to Γ and G with subsequent integration from O to t. In this case according to the Fourier hypothesis it should be supposed that

$$q_v = -\lambda \frac{\partial \Gamma}{\partial v}.$$

Equation (7) has a unique solution which can be found by the method of successive approximations when supposing $\varphi_0 = \Gamma(Q, \mathscr{P}, t)$.

$$\varphi_n(Q, \mathscr{P}, t) = \int_0^t d\tau \int\int_S q_v(Q, M, t - \tau)\varphi_{n-1}(M, \mathscr{P}, \tau)\, dM.$$

Then

$$G = \varphi_0 + \varphi_1 + \varphi_2 + \ldots + \varphi_n + \ldots. \quad (8)$$

Let us prove the uniform convergence of series (8) of which the terms are all positive functions. We have

$$\varphi_n = \int_0^t d\tau \int\int_S q_v \varphi_{n-1}\, dS.$$

But $\varphi_{n-1}(Q, \mathscr{P}, t)$ is the finite function in the interval $0 \leqslant t \leqslant \infty$ at $n \geqslant 2$ since the required type of singularity for G at the point $\mathscr{P} = Q$ and $t = 0$ (at this point it is a delta-function) is provided by the zeroth approximation φ_0. Consequently, $\varphi_{n-1} \leqslant A$. Then $\varphi_n < A \times b$ where

$$b = \max \int_0^t d\tau \cdot \int\int_S q_v\, dS < 1$$

[see (3)]. Applying these estimates to all terms of series (8) we shall get a numerical series which is a geometrical progression with the denominator $b < 1$. This series being a majorant for the functional series (8) converges. Thus series (8) converges uniformly in the interval $0 < t < \infty$. The identity of the terms in series (8) and (6) can be easily shown after the appropriate substitution of variables.

Let us show that in this the condition of heat conservation is obeyed, i.e.

$$k \int\int\int_D G(Q, \mathscr{P}, t)\, d\mathscr{P} = 1 \qquad (Q \in D).$$

Integrate both sides of equation (7) with respect to D

$$k \int\int\int_D G(Q, \mathscr{P}, t)\, d\mathscr{P} = \int_0^t d\tau \int\int_S q_v(Q, M, t - \tau)\, dM \int\int\int_D kG(M, \mathscr{P}, \tau)\, d\mathscr{P} + k \int\int\int_k \Gamma(Q, \mathscr{P}, t)\, d\mathscr{P}.$$

On supposing this condition to be obeyed we come to equality (2) proved above. But the integral equation has a unique solution. Thus the integral

$$k \int\int\int_D G(Q, \mathscr{P}, t)\, d\mathscr{P}$$

which is equal to 1 is defined unambiguously.

4. By the methods indicated in paragraphs 2 and 3 one can find the function $G(Q, \mathscr{P}, t, \tau)$ for the region within the moving boundary. In this case the problem is stated as follows. There appears a heat pulse of unit intensity at some point Q of the region D of variable volume (mass) at the moment τ. The gain of mass (the case of expansion of the region D is considered) has a zero excess temperature. The boundary is heat insulated. Find $G(Q, \mathscr{P}, t, \tau)$. This problem is characterized by the presence of a moving boundary and, hence, by the fact that the temperature

field depends not on the difference $t - \tau$ but on t and τ, since not only the time of action $t - \tau$ but also the moment τ of pulse onset is important.

It should be noted that if the body volume is decreasing steadily and

$$|\cos[r_0(t), v(t)]r_0'(\theta, \varphi, t)| > c$$

(the heat propagation velocity) for all $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$ and $\tau \leqslant \xi \leqslant t_0$, then the temperature field $G(Q, \mathscr{P}, t, \tau)$ will be $\Gamma(Q, \mathscr{P}, t - \tau)$ since the "reflection" flux is excluded together with mass adjacent to the surface. In this case the condition that flux on the boundary equals zero is not obeyed. To obey this condition it is necessary to consider the "vanishing" mass as having zero temperature. In this case the solution to the problem is similar to the solution of the problem of an expanding region, which will be discussed below.

Let us determine $\Gamma(Q, \mathscr{P}, \xi - \tau)$ on a newly appearing bed of the substance, each time representing its zero temperature as

$$\Gamma[Q, M(\xi), \xi - \tau] - \Gamma[Q, M(\xi), \xi - \tau].$$

Thus the first reflection on the element dS for the time $d\xi$ will be the sum of heat, which should pass through dS for the time $d\xi$ due to the heat conduction according to the proper scheme of the problem for the space R (the model is meant of the region expanding in the infinite space of which the temperature field is

$$\Gamma(Q, \mathscr{P}, \xi - \tau)$$

and the region itself does not disturb the process of heat conduction in R), and the enthalpy of substance bed, which has appeared on dS for the time $d\xi$ at the moment ξ due to the motion of the boundary $S(\xi)$ at the temperature

$$\{-\Gamma[Q, M(\xi), \xi - \tau]\}.$$

Therefore, the reflection flux is (4). Now without determination of the reflections of the $1, 2, \ldots n$th order or their effects on the temperature field in D, we shall directly write an integral equation proceeding from the fact that in this problem $G(Q, \mathscr{P}, t, \tau)$ should be considered as the result of superpositions of heat sources with unit intensity q_v, which are placed on S and move together with S, plus the function Γ for the whole space.

$$G(Q, \mathscr{P}, t, \tau) = \int_\tau^t d \iint_{S(\xi)} q_v[Q, M(\xi), \tau, \xi] \times G[M(\xi), \mathscr{P}, t, \xi]\, dM(\xi) + \Gamma(Q, \mathscr{P}, t - \tau). \quad (9)$$

The relation

$$\iint_{S(\xi)} q_v[Q, M(\xi), \tau, \xi]\, dM(\xi) = -k(\partial/\partial\xi) \iiint_{D(\xi)} \Gamma(Q, \mathscr{P}, \xi - \tau)\, d\mathscr{P}$$

holds for q_v. After integrating it with respect to ξ from τ to t we obtain

$$\int_\tau^t d\xi \iint_{S(\xi)} q_v[Q, M(\xi), \tau, \xi]\, dM(\xi) = 1 - k \iiint_{D(t)} \Gamma(Q, \mathscr{P}, t - \tau)\, d\mathscr{P}. \quad (10)$$

Therefore

$$\int_\tau^t d\xi \iint_{S(\xi)} q_v[Q, M(\xi), \tau, \xi]\, dM(\xi) < 1.$$

Using these results it is not difficult to establish the convergence of the series of successive approximations for equation (9) in the same way as was done for equation (7). It should be taken into account that $\varphi_1, \varphi_2, \ldots \varphi_n$ are continuous functions everywhere in $D(t)$ which tend uniformly to zero at $t \to \tau$, since the required singularity can be provided for G at $\mathscr{P} = Q$ and $t = \tau$ by the zero approximation $\varphi_0 = \Gamma(Q, \mathscr{P}, t - \tau)$. Further, the following considerations make this convergence evident. If the heat pulse appears simultaneously both in a body with a fixed boundary and in a similar body with moving surface, which extends its volume, then in the second case the influence function will be less than in the first one since heat diffuses into a larger volume. But for the first problem the convergence of the series is proved, therefore it will be valid for the second case as well, as the series of successive approximations can be visually explained by the "reflection" theory.

The fulfilment of the condition of heat conservation can be proved possible by the well-known method. Integrate both sides of the equation by $D(t)$

$$k \iiint_{D(t)} G(Q, \mathscr{P}, t, \tau)\, d\mathscr{P} =$$

$$\int_\tau^t d\xi \iint_{S(\xi)} q_v[Q, M(\xi), \tau, \xi]\, dM(\xi)$$

$$\iiint_{D(t)} k \cdot G[M(\xi), \mathscr{P}, t, \xi]\, d\mathscr{P} + k \iiint_{D(t)} \Gamma(Q, \mathscr{P}, t-\tau)\, d\mathscr{P}.$$

On supposing

$$\iiint_{D(t)} kG(Q, \mathscr{P}, t, \xi)\, d\mathscr{P} = 1 \qquad [Q \in D(t)]$$

we get true equality (10). But integral equation (9) has a unique solution, therefore

$$\iiint_{D(t)} kG(Q, \mathscr{P}, t, \xi)\, d\mathscr{P} = 1 \qquad (\tau \leqslant \xi \leqslant t).$$

Let us consider in more detail how to obtain the solution of equation (9) by the method of successive approximations. The problem is solved in the spherical system of co-ordinates. On the moving surface the point $M(\xi)$ has co-ordinates $r_0(\theta, \varphi, \xi)$, θ and φ. The surface element in spherical co-ordinates can be written as

$$dS(\xi) = \frac{r_0^2(\theta, \varphi, \xi) \sin\theta\, d\theta\, d\varphi}{\cos[r_0(\xi), \nu(\xi)]}$$

to the accuracy of infinitesimals of a higher order of magnitude. Therefore, on determining q_v from formula (4) we have

$$q_v = q_v[Q(\rho_1, \theta_1, \varphi_1), r_0(\theta, \varphi, \xi), \theta, \varphi, \tau, \xi].$$

Now it is clear how to find $\varphi_n(Q, \mathscr{P}, t, \tau)$ using the recurrent formula. To do this, in

$$\varphi_{n-1}(Q, \mathscr{P}, t, \tau)$$

we should substitute $Q(\rho_1, \theta_1, \varphi_1)$ by

$$M[r_0(\theta, \varphi, \xi), \theta, \varphi],$$

τ by ξ, multiply by

$$q_v[Q(\rho_1, \theta_1, \varphi_1), r_0(\theta, \varphi, \xi), \theta, \varphi, \tau, \xi] \times \frac{r_0^2(\theta, \varphi, \xi) \sin\theta\, d\theta\, d\varphi\, d\xi}{\cos[r_0(\xi), \nu(\xi)]}$$

and integrate by $\varphi(0 \leqslant \varphi \leqslant 2\pi)$, $\theta(0 \leqslant \theta \leqslant \pi)$, and $\xi(\tau \leqslant \xi \leqslant t)$. The problem can also be solved in another system of co-ordinates.

The moving surface will have covered the volume $D(t) - D(\tau)$ bounded by two surfaces $S(\tau)$ and $S(t)$ by the time t, therefore one can determine the reflection series in another way by integrating over the volume

$$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \int_{r_0(\theta, \varphi, \tau)}^{r_0(\theta, \varphi, t)} \ldots \rho^2\, d\rho.$$

In this it should be taken into account that ξ, the time of rise of the reflected pulse at a point of the internal volume, depends on the co-ordinates of this point ρ, θ, φ and is determined from the relation

$$\xi = \xi(\rho, \theta, \varphi).$$

Then $q_v = q_v[Q(\rho_1, \theta_1, \varphi_1), \rho, \theta, \varphi, \tau, \xi(\rho, \theta, \varphi)]$ and to obtain the nth approximation, it is necessary in the $(n-1)$th one to substitute ρ, θ, φ for $Q(\rho_1, \theta_1, \varphi_1)$; $\xi(\rho, \theta, \varphi)$ for τ; multiply by $q_v[Q(\rho_1, \theta_1, \varphi_1), \rho, \theta, \varphi, \tau, \xi(\rho, \theta, \varphi)]$ and find the volume integral.

Since the heat is propagated with finite velocity c, it takes the heat wave front a certain time t_* to catch up the moving point of the surface. t_* can be found from the equation

$$ct_* = r(\theta, \varphi, \tau + t_*);\ t_* = t_*(\theta, \varphi, \tau, c).$$

Thus, the "useful" volume which produces thermal "reflections" is less than stated above and equal to $D(t) - D(\tau + t_*)$. If any point of the surface moves with the velocity of heat wave or with a higher one, the temperature field in D will be the same as in an infinite body. It is easy to show that on determining the reflection flux as in paragraphs 3 and 4, we also take into account the inertness of the process, i.e. its pre-history.

Let D be the region of variable physical properties, i.e. the heat conductivity λ, heat capacity C, specific weight γ depend on $\mathscr{P} \in D$ and t.

After solving for the space R the heat conduction equation with variable coefficients, this equation being obtained from the hypothesis

$$\bar{q}_l = -\lambda(\mathscr{P}, t)\frac{\partial T}{\partial l}$$

(l is an arbitrary direction) provided

$$T\big|_{t=\tau} = \frac{\delta(Q - P)}{k},$$

we find $\Gamma(Q, \mathscr{P}, t, \tau)$. Then to find the function G we can construct an integral equation of the form:

$$G(Q, \mathscr{P}, t, \tau) = -\int_{\tau}^{t} d\xi \iint_{S_t} \lambda(M, \xi)\frac{\partial \Gamma(Q, M, \xi, \tau)}{\partial v} \times G(M, \mathscr{P}, t, \xi)\, dM + \Gamma(Q, \mathscr{P}, t, \tau). \quad (11)$$

The estimate

$$\int_{\tau}^{t} d\xi \iint_{S} q_v\, dS < 1$$

is valid for the reflection flow

$$q_v = -\lambda(M, \xi)\frac{\partial \Gamma(Q, M, \xi, \tau)}{\partial v}.$$

The convergence of the successive approximation series can be easily proved for equation (11) applying the above estimate.

If the problem of substance diffusion is considered, equations (7), (9) and (11) are valid only for a region with a uniformly dissipating boundary, i.e. for such a boundary which reflects uniformly the substance flow in all directions. However the principle of reflection can be applied to the case of a boundary of nonisotropic effect, when the reflected flux is characterized by some dissipation function $\Phi(\theta, \varphi, M)$. The latter need not be a conservative function of the point on the boundary, but can depend on the angle of incidence of the substance and energy carriers to the surface element. This will, probably, allow the solution of some problems on light phenomena, radiation, etc.

5. Solution of the first boundary transfer problem requires determination of the temperature field $T(\mathscr{P}, t)$ in D (the region with a fixed boundary) with the initial condition $T(\mathscr{P}, O)| = f(\mathscr{P})$, boundary condition $T(\mathscr{P}, t)|_S = \bar{\varphi}(\mathscr{P}, t)$ and the source function $F(\mathscr{P}, t)$ in D. For the sake of simplicity suppose that $T(\mathscr{P}, O = 0$ and $F(\mathscr{P}, t) = 0$. Since $G(Q, \mathscr{P}, t)$ can be obtained for D from equation (7), then to solve the problem it is necessary to find from equation (5) the heat flux q which initiates the process.

Let at some moment t the boundary condition be obeyed. To satisfy the required boundary condition at the next moment, the flux consisting of two parts $q = q_1 + q_2$ is necessary for the surface S. The first component of the flux q_1 serves to keep up the same level of the temperature field $\bar{\varphi}(\mathscr{P}, t)$ on the boundary, i.e. it neutralizes its own rate of temperature equalization on the boundary. The second component

$$q_2 = \bar{k}\frac{\partial \bar{\varphi}(\mathscr{P}, t)}{\partial t}$$

is necessary for the required time change of $\bar{\varphi}(\mathscr{P}, t)$, here

$$\bar{k} = \frac{1}{\iint_S G(N, M, O)\, dN}.$$

According to (5)

$$q_1 = -\bar{k}\int_0^t d\tau \iint_S (q_1 + q_2)\frac{\partial G}{\partial t}\, dS.$$

The minus sign allows for the neutralization. By adding

$$\bar{k}\frac{\partial \bar{\varphi}(M, t)}{\partial t}$$

to both sides of the equation, we get

$$q(M, t) = -\bar{k}\int_0^t d\tau \iint_S q(N, \tau) \times \frac{\partial G(N, M, t - \tau)}{\partial t}\, dN + \bar{k}\frac{\partial \bar{\varphi}(M, t)}{\partial t}. \quad (12)$$

In case of non-uniform initial conditions equation (12) is of the form

$$q(M,t) = -\bar{k}\int_0^t \mathrm{d}\tau \iint_S q(N,\tau)\frac{\partial G(N,M,t-\tau)}{\partial t}\mathrm{d}N + \bar{k}\left[\frac{\partial\bar{\varphi}(M,t)}{\partial t} - k\iiint_D f(Q) \times \frac{\partial G(Q,M,t)}{\partial t}\mathrm{d}Q\right]. \quad (13)$$

A strict mathematical demonstration of this fact has probably some difficulties.

The temperature $T(\mathscr{P},t)$ can be obtained from the expression

$$T = \int_0^t \mathrm{d}\tau \iint_S qG\,\mathrm{d}S + k\iiint_D f.G\,\mathrm{d}V,$$

which meets all the conditions of the problem stated. Some physical meaning can be given to the terms of the successive approximations series in equation (12) as it was done in paragraph 3. It is easy to see that q_1 (neutralization flux) is an infinite series of successive compensations. The first compensation is the neutralization of the temperature equalization rate by the flux q_2, the second is the neutralization of the flow influence in the first compensation, etc. As far as the physics of the process is considered, the principle of successive compensations is the only possible means of fulfilling the prescribed boundary condition with the application of the thermal potential

$$\int_0^t \mathrm{d}\tau \iint_S qG\,\mathrm{d}S.$$

The problem for a body with moving boundary (the Stefan problem) is to estimate the temperature field in the region $D(t)$, provided the temperature $\bar{\varphi}(\mathscr{P},t)$ is given on the moving boundary. This problem can be solved when $G(Q,\mathscr{P},t,\tau)$ is known for a body of a variable volume. For this we introduce the unknown heat flux $q(M,t)$ on the moving boundary. The process, essentially, is that of heat propagation from moving sources in a region of variable volume. The condition on the boundary gives

$$\int_0^t \mathrm{d}\tau \iint_{S(\tau)} q(N,\tau)\,G(N,M,\tau)\,\mathrm{d}N(\tau) + k\iiint_{D(0)} f(Q)\,G(Q,M,t,O)\,\mathrm{d}Q = \bar{\varphi}(M,t);$$

$$M[r_0(\theta,\varphi,t),\theta,\varphi] \in S(t);$$

$$N[r_0(\theta,\varphi,\tau),\theta,\varphi] \in S(\tau).$$

Differentiating by t and allowing for the properties of G at $N(t) = M(t)$ and $\tau = t$, transform the above equation of the first kind into the equation of the second kind

$$q(M,t) = -\bar{k}\int_0^t \mathrm{d}\tau \iint_{S(\tau)} q(N,\tau)\frac{\partial G(N,M,t,\tau)}{\partial t}\mathrm{d}N(\tau) + \bar{k}\left[\frac{\partial\bar{\varphi}(M,t)}{\partial t} - k\iiint_{D(0)} f(Q) \times \frac{\partial G(Q,M,t,O)}{\partial t}\mathrm{d}Q\right] \quad (14)$$

The possibility of constructing such a physical model gives some reason to suppose that the successive approximation series for equation (12) is convergent in the problem considered.

which is solved by the method of successive approximations. In both cases the first boundary problem has only one solution.

6. Let us now assume the boundary condition

of the third kind, i.e. that the heat-transfer law and the ambient temperature are specified on the surface. Suppose that the boundary of the body is fixed, the initial condition is uniform and the heat-transfer law is taken in the form of the Newton law of cooling $q = \alpha (T_{sur} - T_{bound})$ (α is the heat-transfer coefficient). Then the equation for q can be obtained as follows. The solution should be found of the form.

$$\int_0^t d\tau \iint_S qG \, dS.$$

Using the boundary condition we get

$$q(M, t) = -\alpha \int_0^t d\tau \iint_S q(N, \tau)\, G(N, M, t - \tau)\, dN + \alpha T_{sur}(M, t). \quad (15)$$

Considering G positive everywhere in D it is easy to show that the series of successive approximations for equation (15) is absolutely and uniformly convergent for the time t from the interval $0 \leqslant t < t_0$, where t_0 is found from the condition

$$\max \int_0^{t_0} d\tau \iint_S G(N, M, t_0 - \tau)\, dN = 1/\alpha.$$

Indeed, we have the estimates

$$|\varphi_0| = |\alpha \,.\, T_{sur}| < A_1 |\varphi_1| \leqslant \alpha \int_0^t d\tau \iint_S |\varphi_0|\, G \, dS < A_1 b_1$$

where

$$b_1 = \alpha \max \int_0^t d\tau \iint_S G(N, M, t - \tau)\, dN < 1.$$

Applying them in succession we get a geometric progression with the denominator $b_1 < 1$, which is a majorizing numerical series for the series $|\varphi_0| + |\varphi_1| + |\varphi_2| + \ldots + |\varphi_n| + \ldots$. Therefore the functional series of successive approximations is absolutely and uniformly convergent.

7. We shall try to show that the class of regions for which the function G is constructed can be extended. Consider a flat region with a diffraction area (Fig. 1). It is seen from Fig. 1 that the line MN, along which the diffraction proceeds, divides the region into two parts $D_1(Q)$ and $D_2(Q)$. It is evident that the regions $D_1(Q)$ and

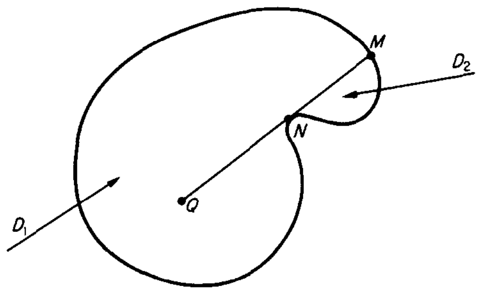

FIG. 1.

$D_2(Q)$ are more convex than the original one. Suppose that G_1 and G_2 are known for the regions $D_1(Q)$ and $D_2(Q)$. $q(\eta, t)$ denotes the flow on the common boundary MN in the direction of the external normal to D_1. Then, on allowing for the continuity of the temperature field in the vicinity of some point L on MN both on D_1 and D_2 sides, we get

$$G_1(Q, L, t) - \int_0^t d\tau \int_M^N q(\eta, \tau) \,.\, G_1(\eta, L, t - \tau)\, d\eta = \int_0^t d\tau \int_M^N q(\eta, \tau) \,.\, G_2(\eta, L, t - \tau)\, d\eta.$$

Transform the written equation of the first kind into an equation of the second kind by differentiating it by t. Similarly considering $D_1(Q)$ and $D_2(Q)$, etc., we reduce the estimation of G by a finite number of procedures to the estimation of the influence functions for the regions which differ little from convex, and to the successive solution of the proper number of integral equations by the usual method. The region is almost convex when it can be transformed into convex by small deformations. Taking account of the fact that small changes of the influence function correspond to small deformations of the region (which can be easily substantiated), one can consider the influence functions found for almost convex regions.

In a similar way G is estimated for multiply connected regions and for those with several diffraction areas. In the latter case the problem is to solve the system of integral equations relative to the unknown fluxes q_1, q_2 on the interface.

The same method, i.e. by consideration of the flow on the contacting surface (contour), allows the solution of the problem of contact heat transfer (the boundary condition of the fourth kind). An ideal contact provides continuity of the temperature field when the common boundary is being passed, and the flow to the region D_1 is the flow to D_2 with an opposite sign. D_1 and D_2 are regions with different physical properties. Using the previous results it is easy to construct an equation for q and thus to solve the problem. To illustrate this we can analyse the problem: the initial temperatures $f_1(\mathscr{P})$, $\mathscr{P} \in D_1$ and $f_2(\mathscr{P})$, $\mathscr{P} \in D_2$ are given for the regions D_1 and D_2 with the common boundary. Find the field of equalization with no contact with the surroundings. G_1 and G_2 are known.

8. Find the source function $G[Q(\xi, \eta, \theta); \mathscr{P}(x, y, z); t]$ for the half-space $R/2$. For this problem we write equation (7).

$$G(Q, \mathscr{P}, t) = \int_0^t \mathrm{d}\tau \iint_{-\infty}^{+\infty} q_\nu(Q, M, \tau) \times G(M, \mathscr{P}, t - \tau)\,\mathrm{d}M + \Gamma(Q, \mathscr{P}, t). \tag{16}$$

Direct Q to some point N on the boundary plane. When $Q \to N$, then $\cos(r, \nu) \to 0$ and, as it is seen from (1), $q_\nu = 0$ for any r which is not zero and for τ from the interval $0 \leqslant \tau \leqslant \tau$ since the surface element $\mathrm{d}M$ is placed in the direction of the vector line of the heat flux. But at $r \to 0 (M \to N)$ and $\tau \to 0$ due to the fact that $\Gamma(N, \mathscr{P}, \tau)$ is delta-shaped, in this point q_ν has such a singularity that

$$\lim_{\substack{r \to 0 \\ \tau \to 0}} q_\nu\,\mathrm{d}M\,\mathrm{d}\tau$$

exists, which is denoted by p. That $p \neq \infty$ is seen from equation (16), since otherwise we should have an incorrect equality between the finite and infinite values. From the same equation it is seen that $p \neq 0$ since $G \neq \Gamma$. Thus by means of limiting transition from the integral equation we get for G an algebraic equation with the unknown coefficient p.

$$G(N, \mathscr{P}, t) = pG(N, \mathscr{P}, t) + \Gamma(N, \mathscr{P}, t),$$

$$G = \frac{1}{1 - p}\Gamma.$$

Integrating both sides of the equation over the half-space, we find p from the condition of heat conservation

$$\iiint_{R/2} kG\,\mathrm{d}V = \frac{1}{1 - p}\iiint_{R/2} k\Gamma\,\mathrm{d}V;$$

$$1 = \left(\frac{1}{1 - p}\right) \cdot \tfrac{1}{2}.$$

Hence $p = \frac{1}{2}$. p is the amount of heat flowing through the solid angle 2π from the point of heat charge onset at the initial moment for infinitely short time. Therefore in this point the heat content becomes zero in as short time as possible. Indeed, this is true since the enthalpy of a point with finite temperature is zero. So $G(N, \mathscr{P}, t) = 2\Gamma(N, \mathscr{P}, t)$. This result agrees with the expression for $G(N, \mathscr{P}, t)$ obtained in the classical theory of unsteady-state heat transfer. The problem above can be referred to as the problem of complete reflection. Now the definition of $G(Q, \mathscr{P}, t)$, where Q is any internal point, leads to the calculation of the integral

$$G(Q, \mathscr{P}, t) = 2\int_0^t \mathrm{d}\tau \iint_{-\infty}^{+\infty} q_\nu(Q, M, \tau) \times \Gamma(M, \mathscr{P}, t - \tau)\,\mathrm{d}M + \Gamma(Q, \mathscr{P}, t).$$

Find $G(\xi, x, t)$ for the finite section l. In this case equation (7) is of the form

$$G(\xi, x, t) = -k\int_0^t \left[\int_0^\xi \frac{\partial \nabla(0, \eta, \tau)}{\partial \tau}\,\mathrm{d}\eta\right] \times G(0, x, t - \tau)\,\mathrm{d}\tau - k\int_0^t \left[\int_0^{l-\xi} \frac{\partial \Gamma(0, \eta, \tau)}{\partial \tau}\,\mathrm{d}\eta\right] G(l, x, t - \tau)\,\mathrm{d}\tau + \Gamma(\xi, x, t). \tag{17}$$

Let $\xi \to 0$ then

$$\lim_{\xi \to 0} \int_0^{\xi} \frac{\partial \Gamma(0, \eta, \tau)}{\partial \tau} \, d\eta = 0$$

for any τ from the half-closed interval $0 < \tau \leqslant t$, since in this interval

$$\frac{\partial \Gamma(0, \eta, \tau)}{\partial \tau}$$

is a finite function. Besides,

$$\lim_{\substack{\xi \to 0 \\ \tau \to 0}} - k\tau \int_0^{\xi} \frac{\partial \Gamma(0, \eta, \tau)}{\partial \tau} \, d\eta = p.$$

This relation proved in the solution of the problem on half-space is also valid for one-dimensional variant. Finally we have

$$G(0, x, t) = -2k \int_0^t \left[\int_0^l \frac{\partial \Gamma(0, \eta, \tau)}{\partial \tau} \, d\eta \right] \times G(l, x, t - \tau) \, d\tau + 2\Gamma(0, x, t).$$

Apply the Laplace transformation by the variable t to both sides of the equation

$$L_t[\varphi(t)] = \int_0^{\infty} e^{-st} \varphi(t) \, dt.$$

On using the procedure of the convolution type

$$L_t \left[\int_0^t \varphi(\tau) f(t - \tau) \, d\tau \right] = L_t[\varphi(t)] \, . \, L_t[f(t)]$$

and taking into account that

$$G(l, x, t) = G(0, l - x, t)$$

due to the symmetry of the functions $G(l, x, t)$ and $G(0, x, t)$ relative to the middle of the section, we obtain

$$f(x, s) = -2k \, \psi(s) f[(l - x), s] + 2f_0(x, s).$$

Here

$$f(x, s) = L_t[G(0, x, t)];$$

$$\psi(s) = L_t \left[\int_0^l \frac{\partial \Gamma(0, \eta, t)}{\partial t} \, d\eta \right];$$

$$f[(l - x) s] = L_t[G(l, x, t)];$$

$$f_0(x, s) = L_t[\Gamma(0, x, t)].$$

Let us present the above functions in the form of series with respect to $\cos n\pi x/l$. This allows the unknown coefficients $a_k(s)$ of the function $f(x, s)$ to be estimated by equating the coefficients preceding $\cos k\pi x/l$ both in the left- and right-hand sides of the equation.

$$f(x, s) = \frac{a_0(s)}{2} + \sum_{k=1}^{\infty} a_k(s) \cos \frac{k\pi x}{l},$$

$$a_k(s) = \frac{2}{l} \int_0^l f(x, s) \cos \frac{k\pi x}{l} \, dx.$$

Find the coefficients $a_k^*(s)$ of the Fourier series of the function $f(l - x, s)$,

$$a_k^*(s) = \frac{2}{l} \int_0^l f(l - x, s) \cos \frac{k\pi x}{l} \, dx.$$

Supposing $l - x = \eta$ we have

$$a_k^*(s) = \frac{2}{l} \int_0^l f(\eta, s) \frac{\cos k\pi(l - \eta)}{l} \, d\eta.$$

But

$$\cos \frac{k\pi(l - \eta)}{l} = \cos k\pi \, . \cos \frac{k\pi\eta}{l} - \sin k\pi \sin \frac{k\pi\eta}{l}; \qquad \cos k\pi = (-1)^k;$$

$$\sin k\pi = 0.$$

Thus $a_k^*(s) = (-1)^k a_k(s)$. Using the inverse Laplace transformation

$$\varphi(t) = \int_{\sigma - i\infty}^{\sigma + i\infty} L_t(s) \, e^{st} \, ds,$$

we find the coefficient $a_k(t)$, as well as $G(0, x, t)$ and $G(l, x, t)$. If the pulse appears inside the section l, then $G(\xi, x, t)$ can be found from equation (17).

Consider another way of obtaining $G(\xi, x, t)$ by means of the reflection theory method (see paragraph 3). It is known that

$$-k\int_0^{\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta$$

and

$$-k\int_0^{l-\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta$$

are the reflections of the heat wave $\Gamma(\xi, x, t)$ (the first reflection) at the moment τ at points $x = 0$ and $x = l$, respectively.

Taking into account that complete reflection occurs at the ends of the bar, we write $G(\xi, x, t)$ in the form of a finite series of successive reflections.

$$G(\xi,x,t) = \Gamma(\xi,x,t) - 2k\int_0^t\left[\int_0^{\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta\right]$$
$$\times\,\Gamma(0,x,t-\tau)\,d\tau - 2k\int_0^t\left[\int_0^{l-\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta\right]$$
$$\times\,\Gamma(l,x,t-\tau)\,d\tau + 2k^2\int_0^t\left[\int_0^{\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta\right]$$
$$\int_\tau^t\left[\int_0^{l}\frac{\partial\Gamma(0,\eta,\theta-\tau)}{\partial\theta}\,d\eta\right]\Gamma(l,x,t-\theta)\,d\theta$$
$$+\,2k^2\int_0^t d\tau\left[\int_0^{l-\xi}\frac{\partial\Gamma(0,\eta,\tau)}{\partial\tau}\,d\eta\right]$$
$$\times\int_\tau^t\left[\int_0^{l}\frac{\partial\Gamma(0,\eta,\theta-\tau)}{\partial\theta}\,d\eta\right]$$
$$\times\,\Gamma(0,x,t-\theta)\,d\theta - \ldots .$$

Since the convergence of the reflection series is good, we may use only the first few terms of the series.

Let us prove the following theorem: if at $t > 0$, Γ has continuous partial derivatives entering the linear differential operator $[P]$ and obeys the equation $[P][\Gamma(Q, \mathscr{P}, t)] = 0$, then at $t > 0$, G which is determined from (7) also has partial derivatives of the same order in D and obeys the equation $[P][G(Q, \mathscr{P}, t)] = 0$. Prove it to be valid, for example, for the case when P is an operator allowing for the finite heat propagation velocity

$$P = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{1}{a^2}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right]. \quad (18)$$

[The Laplace operator Δ is estimated at the point $\mathscr{P}(x, y, z)$].

It is not difficult to show that the integral operator

$$\varphi_n(Q,\mathscr{P},t) = \int_0^t d\tau \iint_S q_v(Q,M,\tau)$$
$$\times\,\varphi_{n-1}(M,\mathscr{P},t-\tau)\,dM$$

has a continuous action on continuous functions. $\varphi_n(Q, \mathscr{P}, O) = 0$ when $\varphi_{n-1}(Q, \mathscr{P}, O) = 0$. Let $\mathscr{P}$ be an internal point of the region D, $Q \in S$, $Q = N$ and $\varphi_0(Q, \mathscr{P}, t) = \Gamma(Q, \mathscr{P}, t)$; then

$$\varphi_1(N,\mathscr{P},t) = \int_0^t d\tau \iint_S q_v(N,M,\tau)$$
$$\times\,\Gamma(M,\mathscr{P},t-\tau)\,dM.$$

The kernel $q_v(N, M, \tau)$ is singular at $\tau = 0$ and $M = N$, but

$$\lim_{t\to 0}\int_0^t d\tau \iint_S q_v(N,M,\tau)\,dM$$

is finite. It equals

$$1 - k\lim_{t\to 0}\iiint_D \Gamma(N,\mathscr{P},t)\,d\mathscr{P} = \tfrac{1}{2},$$

since

$$\lim_{t\to 0}\Gamma(N,\mathscr{P},t) = \frac{\delta(N-\mathscr{P})}{k}.$$

But $\Gamma(M, \mathscr{P}, O) = 0$, thus $\varphi_1(N, \mathscr{P}, O) = 0$ and also $\varphi_2(N, \mathscr{P}, O) = 0$, $\varphi_3(N, \mathscr{P}, O) = 0$, etc. If Q is inside D, then the kernel q_v is finite and the proof of the above statement is trivial. The continuity of the operator is due to the fact that at $\tau \neq 0$, $q_v(N, M, \tau)$ is a continuous function of M and τ.

Form the series

$$\sum_{n=0}^{\infty} \frac{\partial \varphi_n(Q, \mathscr{P}, t)}{\partial t};$$

$$\frac{\partial \varphi_0(Q, \mathscr{P}, t)}{\partial t} = \frac{\partial \Gamma(Q, \mathscr{P}, t)}{\partial t}, \frac{\partial \varphi_n(Q, \mathscr{P}, t)}{\partial t} = \iint_S q_v(Q, M, t) \,.\, \varphi_{n-1}(M, \mathscr{P}, O)\, \mathrm{d}M$$
$$+ \int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau) \frac{\partial \varphi_{n-1}(M, \mathscr{P}, t-\tau)}{\partial t} \mathrm{d}M = \int_a^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau) \frac{\partial \varphi_{n-1}(M, \mathscr{P}, t-\tau)}{\partial t} \mathrm{d}M$$

since $\varphi_{n-1}(M, \mathscr{P}, O) = 0$, $(n = 1, 2, 3, \ldots)$.

Therefore the series

$$\sum_{n=0}^{\infty} \frac{\partial \varphi_n(Q, \mathscr{P}, t)}{\partial t}$$

is formed by the successive application of the above integral operator to the functions beginning with the continuous one (by condition)

$$\frac{\partial \Gamma(M, \mathscr{P}, t)}{\partial t}$$

and is, therefore, an infinite series of continuous functions. Since $\mathscr{P}$ is an internal point, then

$$\left|\frac{\partial \Gamma(M, \mathscr{P}, t)}{\partial t}\right| \leqslant A_2 \qquad (0 \leqslant t < \infty)$$

and the absolute and uniform convergence of the series composed of first derivatives is established in the same way as was done above (see paragraph 3). Hence at $t > 0$, $G(Q, \mathscr{P}, t)$ has a continuous first derivative with respect to t which is equal to

$$\sum_{n=0}^{\infty} \frac{\partial \varphi_n(Q, \mathscr{P}, t)}{\partial t}.$$

Compose the series

$$\sum_{n=0}^{\infty} \frac{\partial^2 \varphi_n(Q, \mathscr{P}, t)}{\partial t^2};$$

$$\frac{\partial^2 \varphi_n(Q, \mathscr{P}, t)}{\partial t^2} = \iint_S q_v(Q, M, t) \,.\, \varphi'_{n-1}(M, \mathscr{P}, O)\, \mathrm{d}M + \int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau) \frac{\partial^2 \varphi_{n-1}(M, \mathscr{P}, t-\tau)}{\partial t^2} \mathrm{d}M.$$

Here

$$\varphi'_{n-1}(M, \mathscr{P}, O) = \left.\frac{\partial \varphi_{n-1}(M, \mathscr{P}, t)}{\partial t}\right|_{t=0}.$$

Let us show that $\Gamma'(Q, \mathscr{P}, O) = 0$ for all $\mathscr{P}$ except $\mathscr{P} = Q$.

$$\Gamma'(Q, \mathscr{P}, O) = \lim_{t \to 0} \frac{\Gamma(Q, \mathscr{P}, t)}{t}$$

but $\Gamma(Q, \mathscr{P}, t) = 0$ for all t less than

$$\frac{r = |Q\mathscr{P}|}{c},$$

since thermal excitation which is propagating at finite velocity c has not yet achieved the point $\mathscr{P}$. Therefore

$$\lim_{t\to 0} \frac{\Gamma(M, \mathscr{P}, t)}{t} = 0.$$

$$\varphi_1'(N, \mathscr{P}, O) = \lim_{t\to 0} \int_0^t \mathrm{d}\tau \iint_S q_v\,(NM, \tau) \frac{\partial \Gamma(M, \mathscr{P}, t - \tau)}{\partial t} \mathrm{d}M = 0$$

because $\Gamma'(M, \mathscr{P}, O) = 0\,(NuM \in S)$ (see above). It is easy to prove that

$$\varphi_2'(M, \mathscr{P}, O) = 0, \ldots \varphi_n'(M, \mathscr{P}, O) = 0$$

by this recurrent procedure. Thus

$$\frac{\partial^2 \varphi_n(Q, \mathscr{P}, t)}{\partial t^2} = \int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau) \times \frac{\partial^2 \varphi_{n-1}(M, \mathscr{P}, t - \tau)}{\partial t^2} \mathrm{d}M,$$

i.e. the series

$$\sum_{n=0}^{\infty} \frac{\partial^2 \varphi_n(Q, \mathscr{P}, t)}{\partial t^2}$$

is formed by successive application of the continuous operator to the continuous (by condition) function

$$\frac{\partial^2 \Gamma(M, \mathscr{P}, t)}{\partial t^2}$$

and is an infinite series of continuous functions.

Since

$$\left|\frac{\partial^2 \Gamma(M, \mathscr{P}, t)}{\partial t^2}\right| \leqslant A_3 \qquad (0 \leqslant t < \infty),$$

the absolute and uniform convergence of the series composed of second derivatives is established as usual. Therefore, at $t > 0$, $G(Q, \mathscr{P}, t)$ has a continuous second derivative with respect to t,

$$\frac{\partial^2 G(Q, \mathscr{P}, t)}{\partial t^2} = \sum_{n=0}^{\infty} \frac{\partial^2 \varphi_n(Q, \mathscr{P}, t)}{\partial t^2}.$$

Similarly one can prove the absolute and uniform convergence of the series

$$\sum_{n=0}^{\infty} \Delta\varphi_n(Q, \mathscr{P}, t),$$

where Δ is the Laplace operator, and thus show double differentiability of $G(Q, \mathscr{P}, t)$ with respect to the co-ordinates x, y, z.

Find $[P]G(Q, \mathscr{P}, t)$ [see equation (18)]. According to what was proved above, the series for G can be differentiated term-by-term as many times as it is necessary.

$$[P]\, G(Q, \mathscr{P}, t) = \sum_{n=0}^{\infty} [P]\, \varphi_n(Q, \mathscr{P}, t)$$

$$[P]\, \varphi_0(Q, \mathscr{P}, t) = [P]\, \Gamma(Q, \mathscr{P}, t) = 0$$

$$[P]\, \varphi_1(Q, \mathscr{P}, t) = [P] \int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau) \times \Gamma(M, \mathscr{P}, t - \tau)\, \mathrm{d}M =$$

$$\int_0^t \mathrm{d}\tau \iint_S q_v(Q, M, \tau)\, [P]\, \Gamma(M, \mathscr{P}, t - \tau)$$

according to the above proof. Thus

$$[P]\, \varphi_1(Q, \mathscr{P}, t) = 0;$$

$$[P]\, \varphi_2 = \int_0^t \mathrm{d}\tau \iint_S q_v\, [P]\, \varphi_1\, \mathrm{d}S = 0,$$

etc. So each term of the series

$$\sum_{n=0}^{\infty} \varphi_n(Q, \mathscr{P}, t)$$

obeys the equation $[P]\,\varphi_n = 0$ and, consequently, the sum of the series $G(Q, \mathscr{P}, t)$ obeys the same equation, too.

Further, it is easy to show that $T(\mathscr{P}, t)$ found from the expression

$$T(\mathscr{P}, t) = \int_0^t \mathrm{d}\tau \int\int_S q(M, \tau)G(M, \mathscr{P}, t - \tau)\,\mathrm{d}M + \int\int\int_D f(Q)G(Q, \mathscr{P}, t)\,\mathrm{d}Q$$

(see above) obeys not only initial and boundary conditions, but also the new heat conduction equation, if the latter is obeyed by $G(Q, \mathscr{P}, t)$.

In conclusion, let us solve the problem with moving heat sources. In a half-plane the boundary contour of which is in contact with radiating medium of steady temperature T_0, a linear heat source is moving along some curve at the velocity $v(t)$. The power q of the source depends on the co-ordinates x and $y = f(x)$ of the curve. The problem considers the ranges of those temperature values when the reverse radiation can be neglected. Otherwise we should deal with non-linear boundary conditions. Initially, the source was at the point x_0, $y_0 = f(x_0)$. Let us find at what time it will be at the point x; $y = f(x)$. Find the length of the curve from x_0 to x; $\mathrm{d}s = \sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2)} = \sqrt{[1 + f'(x)^2]}\,\mathrm{d}x$;

$$s(x) = \int_{x_0}^{x} \mathrm{d}s = \int_{x_0}^{x} \sqrt{[1 + f'(x)^2]}\,\mathrm{d}x;$$

on the other hand

$$s(\tau) = \int_0^\tau \mathrm{d}s = \int_0^\tau v(\eta)\,\mathrm{d}\eta.$$

By equating both expressions one gets $x(\tau)$. The amount of heat generated by the source on the elementary section $\mathrm{d}s$ in the vicinity of the point ξ, which is inside the interval x_0–x, is equal to

$$q[\xi, f(\xi)]\,\mathrm{d}s = q[\xi, f(\xi)]\sqrt{[1 + f'(\xi)^2]}\,\mathrm{d}\xi.$$

On applying the relation $\xi = \xi(\tau)$ we get $q[\xi, f(\xi)]\,\mathrm{d}s = F(\tau)\,\mathrm{d}\tau$. Now the definition of $T(\mathscr{P}, t)$ by the principle of temperature fields superposition does not present any difficulties since $G(Q, \mathscr{P}, t)$ is known

$$T(x, y, t) = \frac{4{\cdot}9}{(1/\varepsilon_1) + (1/\varepsilon_2) - 1}\left(\frac{T_0}{100}\right)^4 \times \int_0^t \mathrm{d}\tau \int_{-\infty}^{+\infty} G(0, \eta; x, y; t - \tau)\,\mathrm{d}\eta + \int_0^t F(\tau)G\{\xi(\tau), f[\xi(\tau)]; x, y; t - \tau\}\,\mathrm{d}\tau.$$

Here ε_1 and ε_2 are the values describing the emissivity of the surroundings and half-space.

The integral method allows the solution of many unsteady-state heat- and mass-transfer problems which can be described by any linear differential or non-differential equations.

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Abstract—The paper offers a method of solving transient linear transfer problems on the assumption that the solution of the Green function for an infinite space is known.

A study of different boundary value problems for a comprehensive class of regions is given; an integral equation is constructed, the solution to which is the influence function for a given region at adiabatically isolated boundary. In the paper it is shown that the constructed integral equation may be solved by the method of successive approximations and the rate of convergence is studied. Further, it is shown that the terms of successive approximation series permit rather lucid and visual physical interpretation facilitating the interpretation of the transfer process.

The construction of the influence function makes it possible to outline in general form the method of solving the first boundary value problem. For this a new integral equation is constructed to find the heat flux providing transfer process at the given boundary conditions, use being made of the influence function for the given region. This integral equation may also be solved by the method of successive approximations which permits a rather clearer physical interpretation.

Résumé—L'article expose une méthode de résolution des problèmes de transport linéaire transitoire, en supposant que la solution du problème du transport de quantité de mouvement est connue pour un espace infini.

On a étudié différents problèmes aux limites pour une classe ètendue de régions. On a établi une équation intégrale, dont la solution est la fonction d'influence pour une région donnée avec une frontière isolée thermiquement. On montre ici que l'équation intégrale obtenue peut être résolue par approximations successives et l'on étudie la rapidité de la convergence. De plus, on montre que les termes de la série des approximations successives permet une interprétation claire facilitant l'interprétation du processus de transport.

La construction de la fonction d'influence rend possible d'esquisser, d'une façon générale, la méthode de résolution du premier problème aux limites. Pour ceci, une nouvelle équation intégrale est établie pour obtenir le flux de chaleur pour les conditions aux limites données, en utilisant la fonction d'influence dans la région donnée. Cette équation intégrale peut être résolue également par approximations successives, ce qui permet une interprétation physique plus claire.

Zusammenfassung—Es wird eine Methode angegeben, die eine Lösung linearer, instationärer Übergangsprobleme gestattet unter der Voraussetzung, dass die Lösung der Bewegungsgleichung für einen unendlich ausgedehnten Raum bekannt ist.

Eine Untersuchung verschiedener Grenzwertprobleme ist für eine grosse Zahl von Bereichen durchgeführt; eine Integralgleichung, deren Lösung die Einflussfunktion für einen gegebenen Bereich bei adiabat isolierter Wand darstellt, wurde eingeführt. Es wird gezeigt, dass die eingeführte Integralgleichung nach der Methode der sukzessiven Approximation gelöst werden kann. Die Konvergenzgeschwindigkeit wird untersucht. Weiter ist gezeigt, dass die Ausdrücke der sukzessiven Näherungsreihe eine klare und anschauliche physikalische Interpretation erlauben, die eine Deutung der Übergangsprozesse erleichtert.

Die Einführung der Einflussfunktion ermöglicht es, die Lösungsmethode für die erste Randbedingung in allgemeiner Form zu umreissen. Dafür ist eine neue Integralgleichung aufgestellt, um den Wärmestrom bei gegebenen Randbedingungen zu erhalten. Diese Integralgleichung kann auch gelöst werden nach der Methode der sukzessiven Approximation. die auch eine klarere physikalische Deutung erlaubt.